W-2/10

EECS 16B Section 4B

Main Topic: Change of Basis

Administrivia:

- HW 4 due Fri, 2/12
- Anonymous Feedback:
  bit.ly/maxwell-16B-feedback-sp21

Agenda:

- Conceptual Review
- Q2: Diagonalization
- Q1: Change of Basis
- Q3: Intro to Inductors

Change of Basis

"What is a basis?" = Set of vectors that "define" a vector space V s.t. every vector $\vec{v} \in V$ can be written as a linear combination of the basis vectors

$$B = \{\vec{b}_1, \vec{b}_2\}$$

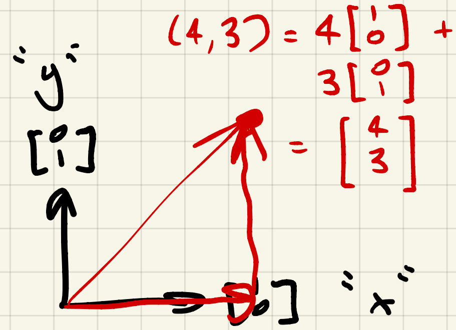
$\forall \vec{v} \in V$,

$$\vec{v} = \alpha \vec{b}_1 + \beta \vec{b}_2$$

= "Our cordinate system"

Standard Basis $\in \mathbb{R}^2$

$$B = \{ \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\vec{b}_1}, \underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\vec{b}_2} \}$$



2D cord. plane

"Can we use a different basis?"

Analogy: Counting with different unit steps

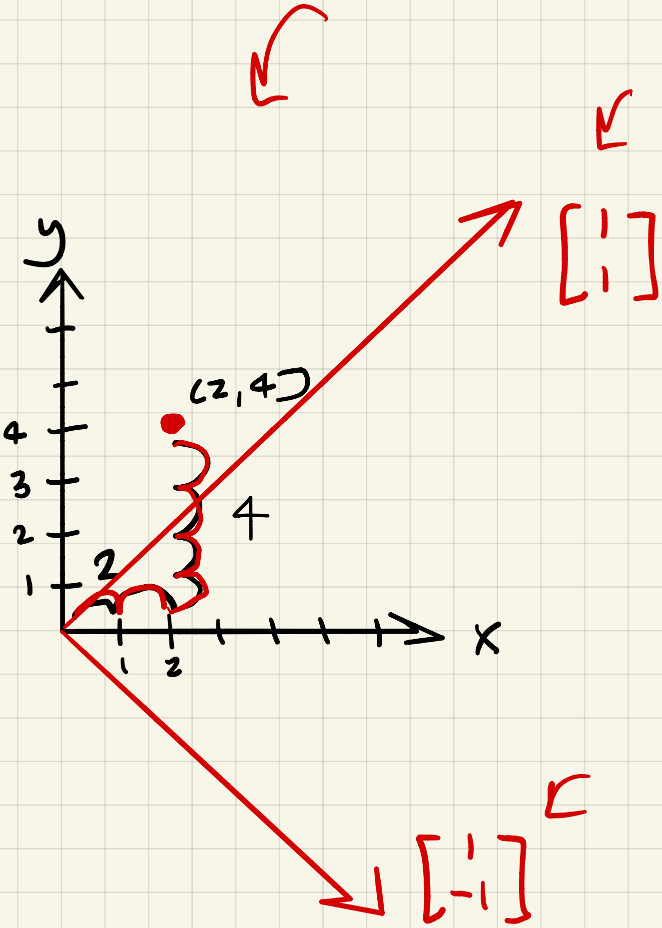
(ex) Concept of "8"

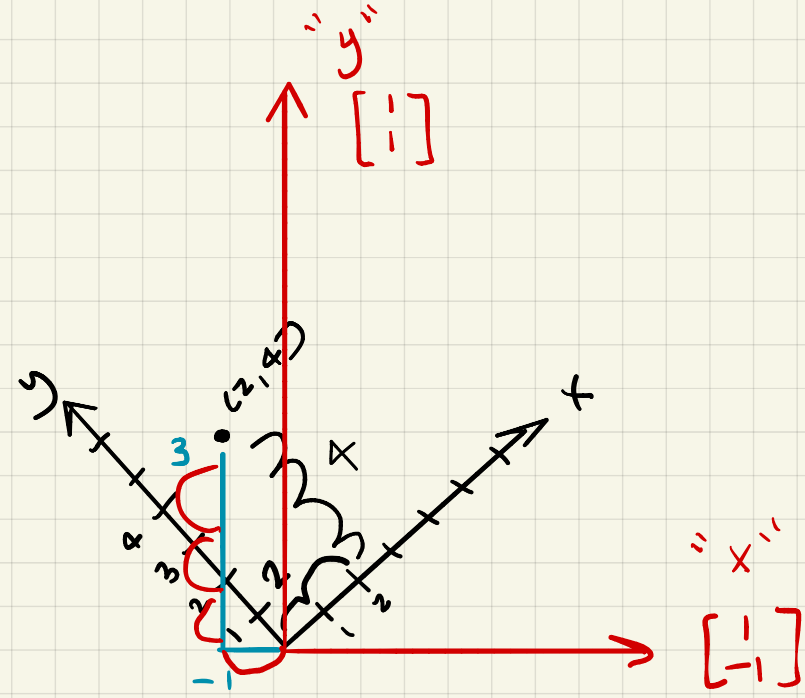
- Units of 1? $[1, 2, 3, 4, \ldots, 8] = 8$ steps $= \frac{8}{1} = 1^{-1} \cdot 8$
- Units of 2? $[2, 4, 6, 8] = 4$ steps $= \frac{8}{2} = 2^{-1} \cdot 8$
- Units of 4? $[4, 8] = 2$ steps $= \frac{8}{4} = 4^{-1} \cdot 8$

Let our unit be U. Then, $8_{new} = U^{-1} \, 8_{old}$

$$\rightarrow \underline{U \, 8_{new}} = 8_{old}$$

ex $\vec{z} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ in standard basis. What is $\vec{w}$, $\vec{z}$'s representation in the basis / coordinate system $P = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$?





$(-1, 3) \rightarrow \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$$\vec{w} = P^{-1} \vec{z}$$

$$P \vec{w} = \vec{z}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

2. Diagonalization

(a) Consider a matrix **A**, a matrix **V** whose columns are the eigenvectors of **A**, and a diagonal matrix **Λ** with the eigenvalues of **A** on the diagonal (in the same order as the eigenvectors (or columns) of **V**). From these definitions, show that

$$\mathbf{AV} = \mathbf{V\Lambda}$$

$A\vec{v} = \lambda\vec{v}$

$A\vec{v}_1 = \lambda_1\vec{v}_1$

$$V = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

AV, $V\Lambda$

$$AV: \quad A\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \\ | & | & & | \end{bmatrix}$$

$A\vec{v}_1 = \lambda_1\vec{v}_1$

$\lambda_1 \quad 0 \quad \cdots \quad 0$

$$V\Lambda: \quad \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$+\begin{pmatrix} \lambda_1\vec{v}_1 & 0 & \cdots & 0 \\ 0 & \lambda_2\vec{v}_2 & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_n\vec{v}_n \end{pmatrix}$$

$$= \begin{bmatrix} | & & | \\ \lambda_1\vec{v}_1 & \cdots & \lambda_n\vec{v}_n \\ | & & | \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$$

$$V = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$$

$$AV = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -2 \\ 10 & -3 \end{bmatrix}$$

$$= \left[\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right]$$

$$= \left[\begin{bmatrix} 8 \\ 10 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 8 & -2 \\ 10 & -3 \end{bmatrix}$$

Diagonalization

$A\vec{v} = \lambda\vec{v}$

WLOG, (Without Loss of Generality) consider an $n \times n$ matrix A with n linearly-independent eigenvalue/eigenvector pairs $(\lambda_i, \vec{v}_i)$.

$$\Lambda = \Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \qquad V = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

From ~~16A/54~~ Above, we know $A\vec{v}_i = \lambda_i \vec{v}_i$

$\to$ $AV = V\Lambda$

Since our eigenvectors are linearly-independent, V is invertible

$$(AV = V\Lambda)V^{-1} \to (A\cancel{V})\cancel{V^{-1}} = (V\Lambda)V^{-1} \to \boxed{A = V\Lambda V^{-1}}$$

"Eigendecomposition" aka "Diagonalization" of A

"Why should we care?"

- "Break apart" A in terms of eigenvalues/eigenvectors
- Easy Matrix Exponentiation $\to$
- Develops idea of "Eigenbasis"

$$A = V\Lambda V^{-1}$$
$$A^2 = (V\Lambda \cancel{V^{-1}})(\cancel{V}\Lambda V^{-1})$$
$$= V\Lambda\Lambda V^{-1}$$
$$= V\Lambda^2 V^{-1}$$
$$A^k = V\Lambda^k V^{-1}$$

Before, "Basis B"

$B = V$

Process:

$$\frac{d}{dt}x = Ax$$

1. Find coefficient matrix A
2. Solve for $A = V \Lambda V^{-1}$
3. $\vec{y} = V^{-1}\vec{z}$; $\vec{z} = V\vec{y}$
4. $\frac{d}{dt}\vec{z} = A\vec{z} = V \Lambda V^{-1}\vec{z}$

   $\frac{d}{dt} V^{-1}\vec{z} = \Lambda V^{-1}\vec{z}$

   $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ $\boxed{\frac{d}{dt}\vec{y} = \Lambda\vec{y}}$ "Decoupled Differential Equations"

5. $\vec{y}(0) = V^{-1}\vec{z}(0)$
6. Solve for $\vec{y}$

   $8_{new} = U^{-1} \cdot 8_{old}$

7. $\vec{z} = V\vec{y}$

"Standard Basis" ↑

↓ "Eigenbasis"

$$\begin{tikzcd}
\vec{z} \arrow[r, "A"] \arrow[d, bend right, "V^{-1}"'] & \frac{d}{dt}\vec{z} \arrow[d, bend right, "V^{-1}"'] \\
\vec{y} \arrow[r, "\Lambda"] \arrow[u, bend right, "V"'] & \frac{d}{dt}\vec{y} \arrow[u, bend right, "V"']
\end{tikzcd}$$

$$\vec{z}\,' = V^{-1}\vec{z}$$

$$\vec{y}\,' = \Lambda V^{-1}\vec{z}$$

$$\vec{y} = V \Lambda V^{-1}\vec{z}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} x_1(t) = -5x_1(t) + 2x_2(t) \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} x_2(t) = 6x_1(t) - 6x_2(t) \tag{2}$$

$$\boxed{\begin{aligned} x_1(0) &= 7 \\ x_2(0) &= 7 \end{aligned}}$$

We can rewrite the above differential equations as a vector differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}t} \vec{x}(t) = A\vec{x}(t), \tag{3}$$

where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $A = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}$. And the diagonalization of A writes

$$A = V\Lambda V^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}. \tag{4}$$

$$\frac{d}{dt}\vec{x} = A\vec{x}$$

$$\frac{d}{dt}\vec{x} = V\Lambda V^{-1}\vec{x}$$

$$\frac{d}{dt}V^{-1}\vec{x} = \Lambda V^{-1}\vec{x}$$

$$\vec{y} = V^{-1}\vec{x}$$

$$\begin{tikzcd}
\vec{x} \arrow[r, "A"] \arrow[d, bend right, "V^{-1}"'] & \frac{d}{dt}\vec{x} \arrow[d, bend right, "V^{-1}"'] \\
\vec{y} \arrow[r, "\Lambda"] \arrow[u, bend right, "V"'] & \frac{d}{dt}\vec{y} \arrow[u, bend right, "V"']
\end{tikzcd}$$

$$\vec{y} = V^{-1}\vec{x}$$

$$\frac{d}{dt}\vec{y} = \Lambda V^{-1}\vec{x}$$

$$\frac{d}{dt}\vec{x} = V\Lambda V^{-1}\vec{x}$$

$$\frac{d}{dt}\vec{y} = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}\vec{y}$$

① Solve for int. cond.

$$x_0 = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$y_0 = V^{-1}x_0$$

$$y_0 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} A_1 e^{-9t} \\ A_2 e^{-2t} \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix}$$

$$x = V\vec{y} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} \, ' \, \end{bmatrix} = \begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -2e^{-9t} + 9e^{-2t} \end{bmatrix}.$$

Standard Basis

$$B = \Big\{ \overset{\vec{e}_1}{\begin{bmatrix}1\\0\end{bmatrix}} \overset{\vec{e}_2}{\begin{bmatrix}0\\1\end{bmatrix}} \Big\}$$

$$\vec{x} = a\vec{e}_1 + b\vec{e}_2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \underline{\mathbf{I}}\vec{x} = \vec{x}$$

"$\underline{V}$" Basis

$$\vec{x} = a_v\vec{v}_1 + b_v\vec{v}_2 = \begin{bmatrix}| & |\\ \vec{v}_1 & \vec{v}_2\\ | & |\end{bmatrix}\begin{bmatrix}a_v\\ b_v\end{bmatrix} = \mathbf{V}\vec{x}_v$$

x_v = representation of x in V Basis

$\underline{x = Vx_v}$ $\underline{x_v = V^{-1}x}$

"U" Basis

$$\vec{x} = a_u\vec{u}_1 + b_u\vec{u}_2 = \begin{bmatrix}| & |\\ \vec{u}_1 & \vec{u}_2\\ | & |\end{bmatrix}\begin{bmatrix}a_u\\ b_u\end{bmatrix} = \underline{\mathbf{U}\vec{x}_u}$$

x_u = representation of x in U Basis

$\underline{x = Ux_u}$ $\underline{x_u = U^{-1}x}$

$$\vec{x} = \begin{bmatrix}| & |\\ \vec{e}_1 & \vec{e}_2\\ | & |\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}| & |\\ \vec{u}_1 & \vec{u}_2\\ | & |\end{bmatrix}\begin{bmatrix}a_u\\ b_u\end{bmatrix} = \begin{bmatrix}| & |\\ \vec{v}_1 & \vec{v}_2\\ | & |\end{bmatrix}\begin{bmatrix}a_v\\ b_v\end{bmatrix}$$

$$\vec{x} = I\vec{x} = \mathbf{V}\vec{x}_v = \mathbf{U}\vec{x}_u$$

$$\begin{tikzcd}
x_u \arrow[r] \arrow[d, bend right, "U"'] & x_u' \arrow[d, bend right, "U"'] \\
x \arrow[r] \arrow[u, bend right, "U^{-1}"'] \arrow[d, bend right, "V^{-1}"'] & x' \arrow[u, bend right, "U^{-1}"'] \arrow[d, bend right, "V^{-1}"'] \\
x_v \arrow[r] \arrow[u, bend right, "V"'] & x_v' \arrow[u, bend right, "V"']
\end{tikzcd}$$

(a) *Transformation From Standard Basis To Another Basis in* $\mathbb{R}^3$

Calculate the coordinate transformation between the following bases:

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{x}_v = \mathbf{T}\vec{x}_u$ where $\vec{x}_u$ contains the coordinates of a vector in a basis of the columns of $\mathbf{U}$ and $\vec{x}_v$ is the coordinates of the same vector in the basis of the columns of $\mathbf{V}$.

Let $\vec{x}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and compute $\vec{x}_v$. Repeat this for $\vec{x}_u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Now let $\vec{x}_u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. What is $\vec{x}_v$?

$\frac{\partial}{\partial t} x_u$

$$\begin{array}{ll} U & x_u \\ & \downarrow \; ? \\ & x \\ & \downarrow \; ? \\ V & x_v \end{array}$$

$$\begin{tikzcd}
x_u \arrow[d, bend right, "U"'] \arrow[rr] & & x_u' \arrow[d, bend right, "U"'] \\
x \arrow[u, bend right, "U^{-1}"'] \arrow[d, bend right, "V^{-1}"'] \arrow[rr] & & x' \arrow[u, bend right, "U^{-1}"'] \arrow[d, bend right, "V^{-1}"'] \\
x_v \arrow[u, bend right, "V"'] \arrow[rr] & & x_v' \arrow[u, bend right, "V"']
\end{tikzcd}$$

$$\vec{x} = U\vec{x}_u$$
$$\vec{x}_v = V^{-1}\vec{x}$$
$$\vec{x}_v = \underline{V^{-1}U}\,\vec{x}_u$$

$$V\vec{x}_v = U\vec{x}_u$$
$$\vec{x}_v = \underline{V^{-1}U}\,\vec{x}_u$$

$$\Rightarrow \quad T = V^{-1}U$$

(b) *Transformation Between Two Bases in* $\mathbb{R}^3$

Calculate the coordinate transformation between the following bases:

$$\mathbf{V} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \mathbf{W} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix},$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{x}_w = \mathbf{T}\vec{x}_v$. Let $\vec{x}_v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and compute $\vec{x}_w$. Repeat this for $\vec{x}_v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Now let $\vec{x}_v = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. What is $\vec{x}_w$?

$$\begin{tikzcd}
x_w \arrow[rr] \arrow[d, bend right, "W"'] & & x'_w \arrow[d, bend right, "W"'] \\
x \arrow[rr] \arrow[u, bend right, "W^{-1}"'] \arrow[d, bend right, "V^{-1}"'] & & x' \arrow[u, bend right, "W^{-1}"'] \arrow[d, bend right, "V^{-1}"'] \\
x_v \arrow[rr] \arrow[u, bend right, "V"'] & & x'_v \arrow[u, bend right, "V"']
\end{tikzcd}$$

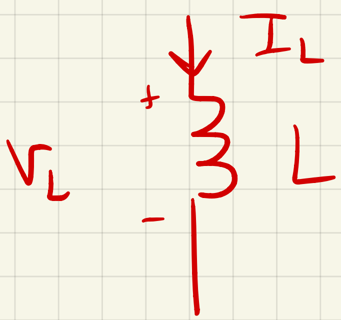
3. Introduction to Inductors

An inductor is a circuit element analogous to a capacitor; its voltage changes as a function of the derivative of the current across it. That is:

$$V_L(t) = L\frac{dI_L(t)}{dt}$$

"free" voltage

$$V_L(t) = L \frac{d}{dt} I_L(t)$$



Capacitors: KCL

$I = C\frac{d}{dt}V$

Inductors: KVL

When first studying capacitors, we analyzed a circuit where a current source was directly attached to a capacitor. In Figure 1, we form the equivalent "fundamental" circuit for an inductor:

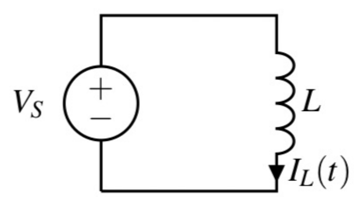


Figure 1: Inductor in series with a voltage source.

(a) What is the current through an inductor as a function of time? If the inductance is $L = 3\text{H}$, what is the current at $t = 6\text{s}$? Assume that the voltage source turns from 0V to 5V at time $t = 0\text{s}$, and there's no current flowing in the circuit before the voltage source turns on.

$$V_L(t) = L\frac{d}{dt}I_L(t)$$

$$\frac{V_S}{L} = \frac{d}{dt}I_L(t)$$

const.

$$I_L(t) = \frac{V_S}{L}t + I_L(0)$$

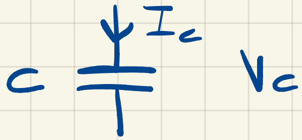
$I_L(6)$:

$\frac{5}{\cancel{3}_{1}}\cancel{6}^{2} + 0$

$= 10$ A.

(b) Now, we add some resistance in series with the inductor, as in Figure 2.
Solve for the current $I_L(t)$ in the circuit over time, in terms of R, L, V_S, t.

Capacitors



$$I_c(t) = C \frac{d}{dt} V_c(t)$$

$$\left(I = C \frac{d}{dt} V \right)$$

C = Capacitance
(measured in Farads F)

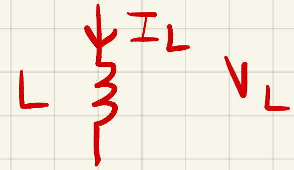
At D.C. / Steady State:

$$I_c(t) = C(0)$$

$$I_c(t) = 0$$

No current passes
=
Infinite Resistance
=
Open Circuit

Inductors



$$V_L(t) = L \frac{d}{dt} I_L(t)$$

$$\left(V = L \frac{d}{dt} I \right)$$

L = Inductance
(measured in Henrys H)

At D.C. / Steady State:

$$V_L(t) = C(0)$$

$$V_L(t) = 0$$

No Voltage Drop
=
No Resistance
=
Short Circuit

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